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FUNDAMENTALS OF MATHEMATICS

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PREFACE

The first dilemma of the authors, when preparing this textbook was its future volume. Most textbooks are thick, with plenty of material, in most cases more than it would be presented in lectures. The reader, either lecturer or student, has to make certain choice for every particular course and to accordingly adjust the order of its presenting. For that reason, fat books usually serve as encyclopedia, where from certain knowledge might be sometimes extracted. Since the Internet can serve the same purpose, who reads fat books these days? For that reason, we have chosen this book to be thin. However, the majority of thin mathematical books provide only cursory information. In most cases, terms and statements are distorted and only short and trivial facts are being proven, more serious omitted or, in the best case, only mentioned due the prejudice that mathematical rigor is not for beginners and non-mathematicians, that they are not able to comprehend more complex statements or their proofs, in these books. In such a manner, mathematics is reduced to usage of prefabricated boring patterns. But the whole truth is that, without thorough insight of the proof details, it is often not possible to achieve neither the full comprehension nor the correct application of mathematics, particularly in other disciplines!

The aim of this book was to rectify these shortcomings. With such an approach this textbook appeared, small in volume, but deep in its contents. It contains fundamental algebraic mathematical knowledge, necessary to students of starting years of mathematics, engineering and applied programs. The explication is direct, rigorous and short. As in a hunt, the „dragon" is hit by the spear, and further „cutting" is left to an intelligent reader! The expected entering level of knowledge is standard understanding of elementary mathematics. It is then solidified here and broadened in a modern and scrupulous way.

This book has 182 pages. The material is divided into nine chapters: Elements of mathematical logic, Set theory, Relations, Functions, Elements of combinatorics, Algebraic structures, Field of real numbers, Field of complex numbers and Elements of linear algebra. The recommended order of presenting the material is just as it is here, although we have tried, wherever possible, to make chapters independent of each other. One semester with two hours of lectures, twice a week, is necessary to cover this material in class.

The book has the following specific qualities, in difference to other textbooks prepared for beginners and non-mathematicians, besides the straightforward style of explication:

- all terms and statements are rigorously and correctly presented;
- many difficult statements were completely proved in the shortest, often original, way, while easier ones are mostly left to the reader as exercises;
- aside to the answer to the standard question „how?", we did our best to answer to „why?", too;
- the material is illustrated using leading, as well as non-standard, examples (different logic and combinatorial problems), which can additionally motivate students, showing that the presented theory is very useful and that mathematical knowledge can be transferred from one area to the other. In such a manner, accompanying examples and exercises that follow, next to the clarification of introduced terms, have the role to make a reader curious and sometimes unexpectedly show applicability of the presented knowledge;
- all important parts of the elementary mathematics are re-introduced here in original, short and strict way, because, in most cases, the previous knowledge base from elementary and higher education (real and complex numbers, quadratic equations, trigonometric functions, roots,...) is relatively poor.

We hope that, besides students, this book will prove useful to lecturers, too. It can serve as an indication how in limited number of lectures all the beauty and usefulness of mathematics could be presented, without loss of its rigor, and yet not „alienating" students from mathematics.

We thank our reviewers, Prof. dr S. Pilipovic, Academician and Prof. dr G. Milovanovic, Academician, as well as, Prof dr S Radjenovic, dr J. Vukmirovic, Prof. Emeritus dr C. Dolocanin, who, through numerous suggestions, substantially influenced the quality of this book. We extend our gratitude to Prof. dr M. Kostic, Rector of the State University of Novi Pazar who contributed to the publishing of the book. We also thank Prof. dr R. Nedeljkovic who, next to translating it into English, also did certain technical and substantial corrections of this manuscript and to the lector Sibela Eminovic.

Novi Pazar,

October 2015

Authors

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ELEMENTS OF MATHEMATICAL LOGIC

1.1. STATEMENT

In all scientific disciplines, as well as in everyday life, the sentences in which something is stated, hence they could be true or false, but not true and false at the same time are of particular important. Such sentences are called **statements**.

Definition 1.1.1.
Every sentence in which something is stated, and that might be true or false, but not true and false at the same time, is called a statement.

The accuracy of certain statement could be verified in different ways, e.g.:

- **facing the related reality**; (mostly used in experimental sciences)
  Today everybody knows that statement „Earth is plain" is not true.
- **using logical analysis** (especially in mathematics);
  That way we may conclude that statement „The square of every real number is a nonnegative number" is true, through the analysis of three possible cases: a real number is positive, negative or a zero. In each of these cases the statement is true.
- **declaring** (especially in mathematics);
  E.g. choosing the axiom „Through a point not on the given line in the given plane there is only one (none, two) line that has no common points with that given line", we differentiate among Euclidean, Riemannian and Lobachevskian geometries.
- **outvoting** (common in social sciences).
  The statement „The indicted is not guilty" is obtained by majority of the jury.
- **using statistical analysis** (frequent in medical sciences);
  E.g. „More baby girls were born" is the statement based on statistical data.

We shall denote statements with small letters of Latin alphabet. These letters we call **statement symbols**. Instead of saying „a statement whose statement symbol is p”, we shall simply say „a statement p”. The fact that statement p is true (false) we shall write as $\tau(p) = \top$ ($\tau(p) = \bot$). We also say that statement p has the **truth value** $\top$ ($\bot$).

Definition 1.1.2.

If p is a statement, then the statement „not p”, denoted by $\neg p$, is called the *negation* of statement p. It is true if and only if the statement p is false.

Therefore, the truth table of the statement ¬p is as follows:

p	$\neg p$
$\top$	$\bot$
$\bot$	$\top$

Definition 1.1.3.

Let p and q be the statements. Then, „p and q”, denoted by $p \wedge q$, is a statement called the *conjunction* of statements p and q. It is true if and only if both p *and* q are true.

The truth table of the conjunction $p \wedge q$ is:

p	q	$p \wedge q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

Definition 1.1.4.

Let p and q be the statements. Then, „p or q”, denoted by $p \vee q$, is a statement called the *disjunction* of statements p and q. It is false if and only if both p and q are false.

The truth table of the conjunction $p \vee q$ is:

p	q	$p \vee q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$

Definition 1.1.5.

Let p and q be the statements. Then the statement, „if p then q", denoted by $p \Rightarrow q$, is a statement called the *implication* of statements p and q. It is false if and only if p is true and q is false.

The truth table of the implication $p \Rightarrow q$ is:

p	q	$p \Rightarrow q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\top$

Last two rows of the table correspond to the Latin saying *ex falso quodlibet* (*from falsity, anything follows*). That way, e.g., the statement „If September has 31 days, then tomorrow is Sunday" is true regardless whether tomorrow is Sunday or not, because September hasn't got 31 days.

Implication $p \Rightarrow q$ might also be read in the following ways:

„from p follows q"

„p is sufficient for q"

„q is necessary for p"

„p only if q".

Definition 1.1.6.

Let p and q be the statements. Then the statement, „p if and only if q", denoted by $p \Leftrightarrow q$, is a statement called the *equivalence* of statements p and q. It is true if and only if statements p and q have the same truth values.

In various statements the phrase „if and only if” is often shortened to „iff”.

The truth table of the statement $p \Leftrightarrow q$ is:

p	q	$p \Leftrightarrow q$
$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\top$

From a set of symbols $p, q, r, \ldots$ (we call them: **statement symbols**), then from symbols of logical operations $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, and from parenthesses we use for separating and defining the order of operations, we build **statement formulae**, following these rules:

1. **Statement symbols** are **statement formulae**.
2. **If** A **and** B **are statement formulae, then** $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \Rightarrow B)$, $(A \Leftrightarrow B)$ **are statement formulae too.**
3. **Statement formulae are derived from finite number of applying rules 1 and 2 in this definition**.

In order to simplify, statement formulae $(A \wedge B)$, $(A \vee B)$, $(A \Rightarrow B)$, $(A \Leftrightarrow B)$ will be denoted simply by $A \wedge B$, $A \vee B$, $A \Rightarrow B$, $A \Leftrightarrow B$ omitting the parenthesses. We also accept that in the sequence $\Leftrightarrow, \Rightarrow, \vee, \wedge, \neg$ every operation symbol has greater power of distinction compared to any other that is to the right from it in this sequence. So e.g. the statement $((p \Leftrightarrow q) \Leftrightarrow ((p \Rightarrow q) \wedge (q \Rightarrow p)))$ is simply rewritten as $(p \Leftrightarrow q) \Leftrightarrow (p \Rightarrow q \wedge q \Rightarrow p)$ and $p \vee q \wedge r$ is a short way to write a statement formula $(p \vee (q \wedge r))$.

1.2. ALGEBRA OF STATEMENTS

Two-member set $\{\top, \bot\}$ and operations $\wedge, \vee, \Rightarrow, \Leftrightarrow, \neg$ on that set defined in the following way:

$\wedge$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

$\vee$	$\top$	$\bot$
$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$

$\Rightarrow$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$

$\Leftrightarrow$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\top$

	$\neg$
$\top$	$\bot$
$\bot$	$\top$

form **a mathematical structure** called **algebra of statements.**

In the algebra of statements, statement formulae that consist of n statement symbols correspond to a n-ary operation (see Ch. 4.3). Every statement symbol may have only one of the values $\top$ or $\bot$. For chosen values of all statement symbols in the certain statement formula, the value of the formula can be calculated in the unique way. So, the statement formula

$$(p \wedge q) \vee (p \Rightarrow q)$$

for truth values of statement symbols p and q, $\tau(p) = \top$, $\tau(q) = \bot$, has the truth value:

$$\tau(\top \wedge \bot) \vee (\top \Rightarrow \bot)) = \tau(\bot \vee \bot) = \bot.$$

Definition 1.2.1.

The statement formula is a tautology if and only if it has the truth value $\top$ for all possible choices of truth values of its statement symbols.

Definition 1.2.2.

Two statement formulae F and G are (semantically) equivalent if and only if $F \Leftrightarrow G$ is a tautology.

Tautologies are important because, among other things, they express **rules of logical inference**. Here are some of them:

Law of the excluded third: $p \vee \neg p$.

Contradiction rule: $\neg(p \wedge \neg p)$.

Double negation law: $\neg(\neg p) \Leftrightarrow p$.

Idempotent laws: $p \wedge p \Leftrightarrow p$, $p \vee p \Leftrightarrow p$.

Associative laws: $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r \quad p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$.

Commutative laws: $p \wedge q \Leftrightarrow q \wedge p$, $p \vee q \Leftrightarrow q \vee p$.

Distributive laws: $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$,

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r).$$

De Morgan's laws: $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$, $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$.

Basic method for analysis whether the given statement formula is a tautology or not, is the **method of truth tables**.

Example 1.2.1. Using truth tables proves that the statement formula

$$\neg(p \Rightarrow q) \Leftrightarrow p \wedge \neg q$$

is a tautology.

Solution.

p	q	$\neg q$	$(p \Rightarrow q)$	$\neg(p \Rightarrow q)$	$p \wedge \neg q$
$\top$	$\top$	$\bot$	$\top$	$\bot$	$\bot$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\bot$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$

Truth values in the last two columns are equal, therefore the given statement formula is a tautology.

Example 1.2.2. Students asked their teaching assistant A what kind of problems they will get for their exam. They got the answer: „If there is a problem from the set theory, it will also be a problem from the mathematical logic." Dissatisfied with the answer, they asked the assistant B the same. And he answered: „There will be a problem from the set theory, or from the mathematical logic". Students then asked their professor the same, and he answered: „At least one of them is telling the truth". Did the professor tell the truth?

Let us denote with p the statement „There will be a problem from the set theory" and with q the statement „There will be a problem from the mathematical logic". The assistant A said: „$p \Rightarrow q$", the assistant B said „$p \vee q$), and the professor's statement was: „$(p \Rightarrow q) \vee (p \vee q)$". Truth tables can prove that the last statement is a tautology, so, professor did tell the truth.

Example 1.2.3. An island is inhabited by two kinds of people: those who always tell the truth, and those who always lie. A foreigner met two locals A and B who, on his question: „What are you?" answered:

A: „B is a truth speaker";

B: „One of us is a truth speaker, and the other one is a liar".

What are these two persons?

Let us introduce statements p: „A is a truth speaker" and q: „B is a truth speaker". Then A said: q, and B said: $(p \wedge (\neg q)) \vee (\neg p \wedge q)$. Since truth speakers always tell the truth, and liars always lie, all four implications:

$$p \Rightarrow q, \quad \neg p \Rightarrow \neg q,$$
$$q \Rightarrow ((p \wedge \neg q) \vee (\neg p \wedge q)), \quad \neg q \Rightarrow \neg((p \wedge \neg q) \vee (\neg p \wedge q))$$

have the truth value $\top$. Let us find values of statement symbols p and q for which it is fulfilled. From the truth table:

p	q	$p \Rightarrow q$	$\neg p \Rightarrow \neg q$	$q \Rightarrow ((p \wedge \neg q) \vee (\neg p \wedge q))$	$\neg q \Rightarrow \neg((p \wedge \neg q) \vee (\neg p \wedge q))$
$\top$	$\top$	$\top$	$\top$	$\bot$	$\top$
$\top$	$\bot$	$\bot$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\top$

follows that all four implications are true if and only if $\tau(p) = \tau(q) = \bot$. That means: A and B are liars.

EXERCISES

1. Fill in the missing words so that the following statements are true:
a) „6 divides an integer n .... 2 divides number n";
b) „6 divides an integer n .... 2 and 3 divide number n";
c) „The triangle is an isosceles triangle .... has at least two equal angles";
d) „....... to be Friday tomorrow in order Wednesday to be yesterday";
e) „It is .... for a number to be greater than 1 in order to be a prime number";
f) „For c to divide ab it is .... that c divides at least one of a and b";
Answers: a) if; b) iff; c) iff; d) necessary and sufficient; e) necessary; f) sufficient.

2. Is this sentence a statement: „This statement is false."
Answer: It is not!

3. Prove that formulae that express the laws of logical inference are tautologies.
Hint: Make the truth tables.

4. Prove that following formulae are tautologies:
a) $(p \wedge q) \Rightarrow p$;
b) $p \Rightarrow (p \vee q)$;
c) $\neg \Rightarrow (p \Rightarrow q)$;
d) $(p \wedge q) \Rightarrow (p \Rightarrow q)$;
e) $\neg(p \Rightarrow q) \Rightarrow p$;
f) $\neg(p \Rightarrow q) \Rightarrow \neg q$.

5. Persons A and B either always speak the truth, or they always lie. What are A and B if A said: „At least one of us is a liar", and B said nothing?
Solution one: Let p be the statement „A is a truth speaker", and q is a statement „B is a truth speaker". A said: $\neg p \vee \neg q$. If he is a liar, then this statement is false. Consequently, statements p and q are true, which could not be the case. So, person A is a truth speaker, p is true and $\neg p \vee \neg q$ is true. But then q is false and, hence, B is a liar.