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Preface

Very complex mathematical ideas are often derived from searching for the answers to the questions that are easy to understand. The *Riemann* zeta function is one of these ideas. The study on this function has arisen from a test related to the distribution of prime numbers [224]. They represent something simple and understandable to almost everyone, but what happens when it comes to their distribution? Finding the next known prime number may seem simple at first glance, but, in practice, it is not always the case. Mathematicians have been looking for a general rule that will dictate the distribution of prime numbers of any size. This search gradually led a mathematician *Bernhard Riemann* to use the complex function theory in order to describe the distribution of prime numbers.

The *Riemann* zeta function plays a central role in many areas in which complex analysis is applied, such as number theory (e.g. generating irrational and prime numbers) [220, 223, 224]. It is also an important tool in signal analysis in many fields of contemporary practice and technology, cryptography. Historically [218, 231], over time, more attention was paid to studying the closed form of the *Riemann* zeta function with positive integer arguments, since such special values dictate the properties of the objects they are associated with. In condensed matter physics, for example, the famous *Sommerfeld* expansion, which is used to calculate the number of particles and the internal electron energy, includes the *Riemann* zeta function with even integer argument values [129]. On the other hand, the spin-spin correlation function of isotropic spin-$1/2$ in the *Heisenberg* model [221] is expressed by $\ln 2$ and *Riemann* zeta function with odd integer arguments [202, 205].

Moreover, the calculation of the *Riemann* zeta function and related series is of relevant importance in computer mathematics [214, 222, 226, 230], with the use of the most advanced software tools such as the Mathematica software package. Although traditional methods are based on the *Euler-Maclaurin* and *Riemann-Siegel* formulas, new techniques and algorithms are constantly being developed [211, 225, 227, 229]. In practice, it is typical for a particular numerical method to be limited to a specific domain. Therefore, when concentrating on the *Riemann* zeta function with odd integer arguments, a special method should be developed to establish a connection between the values of the *Riemann* zeta function with odd and even integer arguments. It is precisely the methods and techniques that have been described and developed in the monograph that offer such an approach and hopefully provide a clear direction towards new results in this field.

The monograph was created as a result of the decades of work of Professor Milorad Stevanović on the study of the *Riemann* ζ function and its application to calculations of various sums and integrals, which can find application in various fields of science and technology [198, 202]. This more than demanding area of complex analysis has prompted the authors to define and prove the features of this function. The necessity of addressing this task has been supported by numerous studies conducted by different authors over the period longer than 200 years. Therefore, the first 3 chapters are devoted to this topic. The authors have made a tremendous effort to provide the reader with a new, clear and innovative way of looking at the most important features of the *Riemann* ζ function. The proofs of the expressed theorems are completely original. These first chapters have also established a good theoretical basis for the subsequent chapters in which the focus of the research is directed towards the problem of calculating multiple sums. The first 2 chapters of the monograph provide an overview of the latest results on the $\zeta(s)$ function while the next 3 chapters present the original contribution of the authors, related to determining the values of integrals and sums in which the *Riemann* function appears.

The first problem that is dealt with in detail refers to calculating the coefficients of the form $F_{(p,q)}(-1)$, $G_{(p,q)}(1)$, $G_{(p,q)}(-1)$, $H_{(p,q)}(-1)$. The significance of this problem, which was first observed by *Euler* and *Goldbach*, is that it refers to the coefficients that are the values of the basic multiple sums of order 2, at the points -1, 1. Some of the multiple sums of order 2, are observed at the point $z = i$. The methods of decomposing expressions into partial fractions, into degrees, by the summation by specially defined subsets of the indices m, n, are used in order to obtain the corresponding functional equations, in the region $|z| \leqslant 1$ and especially at $|z| = 1$. Such an approach is completely unique, as well as the techniques used to obtain the required results, which are therefore completely original. The functions $F_{(p,q)}(x)$, $G_{(p,q)}(x)$, $H_{(p,q)}(x)$ for $|x| \leqslant 1$ have been derived in order to establish various functional relations between them and specially, the relations for $x = -1$, 1. Based on the procedure that has been conducted in the monograph, one can conclude that the results are obtained independently of the results achieved by *Nielsen* and that they are more general when compared to them. In addition, some special cases of the above mentioned coefficients, which in this form have not existed in the professional mathematical literature, are provided.

For example, for the coefficients mentioned above the solution has been found when $p+q$ is an odd number (Chapter 4). Moreover, some special cases of the above coefficients, when $p+q$ is even number, are also provided in the monograph. In addition, certain relations that are inverses of each other are established. All 16 $\omega_j(p,q)$ coefficient values have been calculated, 10 of them when $p+q = 2k+1$ and 6 when $p+q = 2k$. At the end of that chapter, the formulas are presented for:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^r} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right)$$

for $r = 1$ and for $r = 2k$. Among other things, the value of the sum:

$$\sum_{n=1}^{\infty} \frac{1}{n^r} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right),$$

has been defined, providing the generalization of one of the *Hardy*'s formulas from the 20s of XX century. The analysis of these sums has also led to some innovative and interesting combinatorial formulas. What remains as an open problem, after the results have been obtained, refers to the cases complementary to the solved ones.

Chapter 5 presents the integration of the function $f(z) = \frac{\log^2(1-z)}{z}$ along the appropriate contour, in order to obtain certain generator relations for multiple summation. These generator relations are very suitable for successive differentiation and integration. The summation formulas have been obtained in which $\zeta(3)$ and *G-Catalan* constant appear, while based on the determined value for $G_{(2k,1)}(-1)$ (in Chapter 4) the relation has been obtained for:

$$\sum_{m=1}^{\infty} \frac{\cos m\alpha}{m^{2k}} \sum_{n=1}^{\infty} \frac{1}{n}, 0 \leqslant \alpha \leqslant \pi,$$

and then the formula for $\alpha = \frac{\pi}{3}$. It has been proven that with the application of purely combinatorial considerations, triple sums can be reduced to double sums. Some of the formulas are just listed without proof, when the authors were aware of the possibilities and generalizations. The contour integration method has been used in this chapter, combined with the method of applying various combinatorial relations. From the basic triple sums in the following form:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{(-1)^{\lambda m+\mu n+\nu p}}{mnp(m+n+p)}, \quad \lambda, \mu, \nu \in \{0, 1\}$$

the formulas have been obtained for all four possible sums (possible values of the parameters λ, μ, ν). These sums are directly related to the integrals:

$$\int_0^1 \frac{\log^3(1-x)}{x} \,\mathfrak{d}\, x, \quad \int_0^1 \frac{\log^3(1+x)}{x} \,\mathfrak{d}\, x$$

so that their values can be determined based on the formulas already obtained in Chapter 4. The formulas for all the sums in the form of:

$$\sum_{m=1}^{\infty} \frac{\lambda^m}{m^2} \sum_{n=1}^{\infty} \frac{u^n}{n} \sum_{p=1}^{\infty} \frac{\nu^p}{p}, \quad \lambda, u, \nu \in \{-1, 1\}$$

have also been obtained. Various combinatorial sums and the results from Chapter 4 have been used in the calculation of these sums. Finally, in Chapter 5, the multiple sums:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{1}{mnp(m+n+p)^3}, \quad \sum_{m=1}^{\infty} \frac{1}{m^a} \sum_{n=1}^{\infty} \frac{1}{n^b} \sum_{p=1}^{\infty} \frac{1}{p}$$

for $(a, b) = (3, 1)$ and for $(a, b) = (2, 2)$ have also been calculated. The problems that remain open are the following:

1) To derive the formula for the sum:

$$\sum_{m=1}^{\infty} \frac{\lambda^m}{m^a} \sum_{n=1}^{\infty} \frac{\mu^n}{n^b} \sum_{p=1}^{\infty} \frac{\nu^p}{p^c}, \quad \lambda, \mu, \nu \in \{-1, 1\}$$

in the case when $a + b + c = 2k$ and in the case when $a + b + c = 2k + 1$.

2) To derive the formula for the sum:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{(-1)^{\lambda m+\mu n+\nu p}}{m^a n^b p^c (m+n+p)^d}, \quad \lambda, \mu, \nu \in \{0, 1\}$$

in the case when $a + b + c + d = 2k$ and in the case when $a + b + c + d = 2k + 1$.

While analyzing the aforementioned problems, one of the basic problems when working with multiple sums has also been tackled: how to express the sums of s multiplicity over the sums of r multiplicity ($r \leqslant s-1$). The monograph offers some partial results without providing a complete answer to this question. The values of the *Riemann* ζ function appear in all of the obtained results. In some formulas, the expressions of the type $\beta_r(-1)$ and $G_{(2,2)}(-1)$ also appear. It is natural to expect that when $\beta_{2r}(-1)$ does not already have a formula that allows the calculations over already known values, among the sums whose multiplicity is 2, there is also the value with such characteristic. However, the obtained results provide the basis, in various expressions given over $G_{(2,2)}(-1)$, for the opposite opinion as well. This opens the problem of determining the base set of elements for the sums of corresponding multiplicity.

Chapter 5 discusses the *Mordell* sums of the following form:

$$S_n = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{1}{r^n s^n (r+s)^n}, \quad n \in \mathbb{N}$$

For $n = 2k$, the above formula was obtained by *Subbarao* and *Sitaramachandrarao* in 1985. The formula for S_{2k+1} has been generally unknown so far, while for S_1 the formula was familiar even to *Euler*. Based on some of the basic results cited in the first chapters of the monograph, the authors have come up with the formulas for S_3 and S_5. Based on the results and formulas derived in Chapter 4, the formula for each S_{2k+1} has been obtained, and the formula for S_n has also been provided. In relation to the result of *L. Tornheim*: "S_{2k+1} is a polynomial of $\zeta_2(1)$, $\zeta_3(1), \ldots, \zeta_{6k+3}(1)$ with rational coefficients", it can be deduced from the derived relations for S_{2k+1} that S_{2k+1} is a 2^{nd} degree polynomial in relation to $\zeta_2(1), \zeta_3(1), \ldots, \zeta_{6k+3}(1)$ with integers. In this chapter, the sum:

$$W(m,n,p) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{(-1)^{\lambda r+\mu s}}{r^m s^n (r+s)^p}, \quad \lambda,\mu \in \{0,1\}\,, \quad m+n+p = 2k+1$$

has also been calculated, giving a generalization of the *Mordell* sums and sums similar to them. In order to obtain the above results, the method of calculating finite combinatorial sums and method of their double summation have been used. In all the cases, complementary sums appear which have enabled us to obtain elegant analytical expressions.

The following Chapter 6 of the monograph is devoted to the double sums of the following form:

$$\sum_{m=1}^{\infty} \frac{\lambda^m}{r^\alpha} \sum_{n=1}^{m} \frac{\mu^n}{s^b}, \quad a+b=4, \quad \lambda,\mu \in \{-1,1\}\,, \quad r \in \{m, 2m-1\}\,, \quad s \in \{n, 2n-1\}\,.$$

In total, there are 40 of these sums and they are of the same form as those considered in Chapter 4. The need to determine these sums lies in the fact that they represent the first non-trivial case in which $p+q = 2k$. Not all the formulas are derived, while in addition to the ones that have been obtained, the relations between some of the given sums have been established. It can be noticed that $\beta_4(-1)$ and $G_{(2,2)}(-1)$ appear in the expressions representing the corresponding sums. The following question naturally arises: what is the relation between $\beta_4(-1)$ and $G_{(2,2)}(-1)$? The formulas derived have been obtained by multiplying specially selected degree series and combining the corresponding double sums with certain products.

Chapter 6 presents various functional relations in the region: $-1 \leqslant x \leqslant 1$, or in one of its sub-regions. The starting point for deriving these functions was the *Spencer* relation [17], which defines the expression for $\zeta_3\left(\frac{x}{x-1}\right)$ in the region $0 \leqslant x \leqslant \frac{1}{2}$. The resulting expressions for $G_{(2,1)}(x)$, $G_{(1,2)}(x)$ in the region $0 \leqslant x \leqslant \frac{1}{2}$ are simple examples that illustrate this. For the same functions we also have corresponding decompositions in the regions $0 \leqslant x \leqslant 1$, $0 \leqslant x < 1$. This is followed by the formulas for the integrals of the following form:

$$\int_0^x \frac{\log^k(1-t)}{t}dt, \quad \int_0^x \frac{\log^k(1+t)}{t}dt, \quad k = 1,2,3$$

in various regions, with decompositions of the functions $G_{(2,2)}(x)$, $G_{(3,1)}(x)$ in the region $0 \leqslant x \leqslant \frac{1}{2}$. For $\zeta_k\left(\frac{1}{2}\right)$ the formulas for $k = 1,2,3$ are known. The relations between $\zeta_4\left(\frac{1}{2}\right)$ and $G_{(2,2)}(-1)$ are presented. This provides the basis for the claim that $G_{(2,2)}(-1)$ can be expressed over $\log^4 2$, $\zeta_2(1)\log^2 2$, $\zeta_3(1)\log 2$ and $\zeta_4(1)$. This is supported by the formulas for some integrals, in which $G_{(2,2)}(-1)$ appears. The values for the expression $G_{(p,q)}\left(\frac{1}{2}\right)$ have been calculated, when $p+q=4$. For the following functions (given by triple sums):

$$G_{(a,b,c)}(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^a} \sum_{n=1}^{m} \frac{1}{n^b} \sum_{p=1}^{n} \frac{1}{p^c}, \quad a+b+c=4$$

the formulas in the region $-1 \leqslant x \leqslant \frac{1}{2}$ have been obtained, as well as the following formulas:

$$G_{(2,1,1)}(x) = -G_{(1,3)}\left(\frac{x}{x-1}\right), \quad G_{(1,2,1)}(x) = -G_{(2,2)}\left(\frac{x}{x-1}\right)$$

$$G_{(1,1,2)}(x) = -G_{(3,1)}\left(\frac{x}{x-1}\right), \quad -1 \leqslant x \leqslant \frac{1}{2}.$$

The following formula is true in the same region:

$$G_{(1,1,\ldots,1)_r}(x) = -\zeta_r\left(\frac{x}{x-1}\right), \quad r \geqslant 2.$$

Furthermore, in Chapter 6, for the functions:

$$F_r(x) = \sum_{k_1=1}^{\infty} \frac{x^{k_1}}{k_1} \sum_{k_2=1}^{k_1} \frac{x^{k_2}}{k_2} \cdots \sum_{k_r=1}^{k_{r-1}} \frac{x^{k_r}}{k_r}, \quad r \geqslant 2,$$

$$F_0(x) = 1, \quad F_1(x) = \zeta_1(x), \quad -1 \leqslant x < 1,$$

one functional and, as a consequence, one recurrent relation have been obtained. The values of $F_r(x)$ have been specified when $r = 0, 1, \ldots, 6$. The problem remains to determine the formula for $\zeta_k(\frac{1}{2})$, $k \geqslant 4$ (if we do not think that the derived formula is true when $k = 4$). The results reported in this chapter have been obtained by combining the combinatorial summation method and multiple summation methods and using various functional relations for the functions given either through multiple sums or by integrals. The method used for proving the functional relation for $F_r(x)$ could be named the "entry-exit" method for multiple sums.

Based on the results obtained by *H. M. Srivastava* in his study from 1988, related to several different groups of summation formulas with the series in which the *Riemann* zeta function appears (which was first investigated by *Euler* and *Goldbach*), in Chapter 7 of the monograph the authors have dealt with the series of this type. For each of the formulas listed in the introduction to this chapter, a formula is given that generalizes the result and that is valid for the natural number n. The generalizations are related to the following sums (for which the appropriate formulas are indicated):

$$\begin{array}{llll} \displaystyle\sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{k+n}, & n \geqslant 1, & \displaystyle\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k+n)}, & n \geqslant 1, \\ \displaystyle\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)-1}{k+n}, & n \geqslant 1, & \displaystyle\sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{2k+2n+1)}, & n \geqslant 0, \\ \displaystyle\sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{k+n}, & n \geqslant 1, & \displaystyle\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{2^k(k+n)}, & n \geqslant 1. \end{array}$$

The problem that remains open refers to the calculation of the following sum:

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)-1}{(k+n)^r}, \quad r \geqslant 2,\ r, n \in \mathbb{N}.$$

In order to obtain the results, the authors used the method of integrating the function which is the product of the polynomial and $\log \Gamma(x)$, where for $\log \Gamma(x)$ the corresponding *Kumer* series in the region $0 < x < 1$ is used. Common to all the problems discussed in the monograph is that all of the multiple sums and integrals are expressed over the *Riemann* zeta function. While this presents only a tiny fraction of what can be calculated, it certainly is one of the irrefutable proofs that this function is of exceptional importance in the theory of multiple series summation.

The following notation was used throughout the text of the monograph: for positive A, label $B = O(A)$ (which is the same as $B \ll A$) indicates that there is an absolute positive constant c so that $|B| \leq cA$. In addition, everywhere throughout the monograph log was used instead of ln. In the sums by nontrivial zeros of the $\zeta(s)$ function, the zeros are numbered in order of the absolute magnitudes of their imaginary parts, and if the absolute values of the imaginary parts are the same, then the order is arbitrary.

Finally, the sad fact is that Professor Stevanović died in 2010 which faced the co-author, who was his close associate and colleague, with a huge and at times hard-to-overcome problem how to adequately describe and expand the results that professor Stevanović achieved during his extremely fruitful life. The co-author can only hope that his effort and desire have resulted in a high-quality work and that the monograph in the form in which it is now will find its way to a professional reading audience.

The reviewers, Professor Dragomir Simeunović and Professor Milan Tasković, with their extensive knowledge and experience, as well as well-meaning suggestions, have provided tremendous help and support in the process of preparing the monograph, for which we are very much obliged.

This text would have probably never seen the light of day that it hadn't been for the wife of Professor Stevanović, Vera, who prepared a huge piece of material which had been in the form of handwritten notes for further revision and computer processing. Mladen Janjić, a late professor's student, did the fracture and prepared the monograph for printing, which is why we would like to express our sincere gratitude to him.

Furthermore, the authors would like to thank Lena Tica for proofreading and editing the English version of the manuscript.

Notation

Owing to the nature of this text, absolute consistency in notation could not be attained, although whenever possible standard notation is used. Notation used commonly through the text is explained there, while specific notation introduced in the proof of a theorem or lemma is given at the proper place in the body of the text.

$k, l, m, n, \ldots$ – Natural numbers (positive integers).

p – A generic prime number.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ – The sets of natural numbers, integers, real and complex numbers, respectively.

$A, B, C, C_1, \ldots$ – Absolute, positive constants (not necessarily the same ones at each occurrence).

ε – An arbitrarily small positive number, not necessarily the same one at each occurrence.

s, z, w – Complex variables ($\mathfrak{Re}\, s$ and $\mathfrak{Im}\, s$ denote the real and imaginary part of s, respectively; common notation is $\sigma = \mathfrak{Re}\, s$ and $t = \mathfrak{Im}\, s$).

t, x, y – Real variables.

$\operatorname*{res}_{s=s_0} F(s)$ – Denotes the residue of $F(s)$ at the point $s = s_0$.

$\zeta(s)$ – The *Riemann* zeta-function is defined with $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\mathfrak{Re}\, s > 1$ and otherwise by analytic continuation.

$\Gamma(s) = \int_0^{\infty} x^{s-1} \mathfrak{e}^{-x}\, \mathfrak{d}x$ for $\mathfrak{Re}\, s > 0$, otherwise by analytic continuation by $s\Gamma(s) = \Gamma(s+1)$. This is the *Euler* gamma-function.

γ – *Euler's* constant $\gamma = -\Gamma'(1) = 0.5772157\ldots$

$\chi(s)$ – The function defined by $\zeta(s) = \chi(s)\zeta(1-s)$, so that by the functional equation for $\zeta(s)$ we have $\chi(s) = \dfrac{(2\pi)^s}{2\Gamma(s)\cos(\frac{\pi s}{2})}$.

$\theta(t)$ – For real t defined as $\theta(t) = \mathfrak{Im}\left\{\log\Gamma(\frac{1}{4} + \frac{1}{2}\mathfrak{i}t)\right\} - \frac{1}{2}t\log\pi$.

$\rho = \beta + \mathfrak{i}\gamma$ – A complex zero of $\zeta(s)$; $\beta = \mathfrak{Re}\,\rho$, $\gamma = \mathfrak{Im}\,\rho$.

$N(T)$ – The number of zeros $\rho = \beta + \mathrm{i}\gamma$ of $\zeta(s)$, counted with multiplicities, for which $0 < \gamma \leq T$.

$N(\sigma, T)$ – The number of zeros ρ of $\zeta(s)$ for which $\beta \geq \sigma$, $|\gamma| \leq T$.

$S(T) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + \mathrm{i}T\right)$.

$\mu(\sigma)$ – For real σ defined as $\mu(\sigma) = \limsup_{t\to\infty} \dfrac{\log|\zeta(\sigma + \mathrm{i}t)|}{\log t}$. The *Mobius* function, defined as $\mu(n) = (-1)^k$ if $n = p_1 \cdots p_k$ (the p_j's being different primes) and zero otherwise, and $\mu(1) = 1$.

$\exp(z) = \mathfrak{e}^z$.

$\mathrm{e}(z) = \mathfrak{e}^{2\pi \mathrm{i} z}$.

$\log x = \log_{\mathfrak{e}} x \equiv \ln x$.

$[x]$ – The greatest integer not exceeding the real number x.

$\{x\} = x - [x]$, the fractional part of x.

$\sum_{n\leq x} f(n)$ – A sum taken over all natural numbers n not exceeding x; the empty sum is defined to be equal to zero.

$\sum_{d|n}$ – A sum taken over all positive divisors of n.

$\Lambda_k(n)$ – The generalized *von Mangoldt* function defined by $\Lambda_k(n) = \sum_{d|n} \mu(d) \left(\log \frac{n}{d}\right)^k$; $\Lambda_1(n) = \Lambda(n)$, the ordinary *von Mangoldt* function.

$\prod_j$ – The product taken over all possible values of the index j; the empty product is defined to be unity.

$\psi(x)$ – Equals $x - [x] - \frac{1}{2}$, or $\psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$.

$\psi_k(x) = \sum_{n\leq x} \Lambda_k(n)$.

$\pi(x) = \sum_{p\leq x} 1$, the number of primes not exceeding x.

$\theta(x) = \sum_{p\leq x} \log p$.

$M(x) = \sum_{n\leq x} \mu(x)$.

$r(n)$ – The number of ways n can be written as a sum of two integer squares.

$d_k(n)$ – The number of ways n can be written as a product of $k \geq 2$ fixed factors; $d_2(n) = d(n)$ is the number of divisors of n.